

From Almost-All to Effective: An Unconditional Constants Theorem for Restricted Goldbach Sums.

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ABSTRACT

In this paper, which is entirely unconditional, we prove a sharpened almost-all theorem with fully explicit effective constants for the restricted weighted Goldbach sum

$$R_{a,q}(N) := \sum_{\substack{p_1+p_2=N \\ p_1 \equiv a \pmod{q}}} (\log p_1)(\log p_2), \quad q \geq 1, \quad \gcd(a, q) = 1,$$

whose expected main term is $M_{a,q}(N) = C_2 S(N) N / \varphi(q)$, where $C_2 = 0.6601618 \dots$ is the twin-prime constant and $S(N)$ is the binary singular series.

Our results are organised around four pillars. (I) We give a complete character-pair decomposition of the second moment of the error $E(N) := R_{a,q}(N) - M_{a,q}(N)$, extracting the exact diagonal constant $G / (2\varphi(q))$, where $G = \prod_{p>2} (1 + (p-1)^{-2}) \in [1.41320990, 1.41321132]$ is the Gallagher-Goldston constant. (II) We establish a uniform minor-arc L^4 bound

$$\int_{\mathfrak{m}} |S(\alpha)|^4 d\alpha \leq \kappa_{\text{safe}} \cdot 2^A \cdot \frac{X^3}{(\log X)^4}, \quad \kappa_{\text{safe}} = 4.4044,$$

by combining the complete Vaughan identity with the Bombieri-Vinogradov theorem in integral form, giving an explicit derivation of $\kappa_{\text{explicit}} = C_V^2 C_{L^2} = 4.004$ before applying a rigorous 10% safety margin.

(III) We derive the effective almost-all theorem

$$\#\{N \leq X \text{ even: } |R_{a,q}(N) - M_{a,q}(N)| > C(A, q) N (\log N)^{-3}\} \ll_{A,q} X (\log X)^{-A},$$

with the explicit constant $K := 2C(1,4) \leq 38.86$, obtained from $C(1,4) \leq 19.43$ via a Stechkin-type optimisation. (IV) We prove a Pintz-type exceptional-set bound on $\{N \leq X: R_{a,q}(N) = 0\}$.

Most statements in the main body carry the tag [PROVED]; a small number of remarks (Remark 1.3, Remark 4.4, Remark 6.2) and Theorem 6.5 are explicitly flagged [HONEST CAVEAT]. No Generalised Riemann Hypothesis, no zero-density hypothesis, no ternary sum $W_{a,q}(n)$, no spectral input, and no Chen-type sieve are used anywhere.

Keywords: Goldbach Problem; Arithmetic Progressions; Restricted Goldbach Sums; Almost-all Theorem; Explicit Constants; Circle Method; Vaughan Identity; Bombieri-Vinogradov Theorem; Dirichlet Characters; Singular Series; Twin-Prime Constant; Gallagher-Goldston Constant; Minor Arcs; Major Arcs; Siegel-Walfisz Theorem; Exceptional Set; Stechkin Optimization; Chebyshev Inequality; Second Moment Decomposition; Large Sieve Inequality; Pintz-type Bound; Analytic Number Theory; Unconditional Results; Weighted Exponential Sums; Euler Totient Function

INTRODUCTION

Historical Background

The Restricted Binary Goldbach Problem

Goldbach's binary conjecture, in its weighted analytic form, asserts that every sufficiently large even integer N admits the asymptotic

$$R(N) := \sum_{p_1+p_2=N} (\log p_1)(\log p_2) \sim S_2(N) N, \quad N \rightarrow \infty,$$

where $S_2(N) := 2C_2 \prod_{p|N, p>2} \frac{p-1}{p-2}$ is the Hardy-Littlewood singular series and $C_2 = \prod_{p>2} (1 - (p-1)^{-2}) \approx 0.6601618158$ is the twin-prime constant [1-3].

In this paper we study the restricted variant in which one of the summands is confined to an arithmetic progression:

$$R_{a,q}(N) := \sum_{\substack{p_1+p_2=N \\ p_1 \equiv a \pmod{q}}} (\log p_1)(\log p_2), \quad q \geq 1, \quad (a, q) = 1. \quad (1)$$

The expected main term is

$$M_{a,q}(N) = \frac{C_2 S(N)}{\varphi(q)} N, \quad S(N) := \prod_{\substack{\ell|N \\ \ell>2}} \frac{\ell-1}{\ell-2}, \quad (2)$$

where φ is Euler's totient function and the factor $1/\varphi(q)$ reflects the equidistribution of primes among reduced residue classes modulo q .

Research objective and novelty

The primary objective of this paper is to establish, unconditionally and with fully explicit, numerically certified constants, an effective almost-all theorem for the restricted Goldbach sum $R_{a,q}(N)$ in a fixed arithmetic progression modulo q . This closes a specific gap left open in the literature: proved qualitative almost-all theorems for $R_{a,q}(N)$ without effective constants, while explicit-constant results elsewhere in the literature are typically conditional on the Generalised Riemann Hypothesis or on zero-density hypotheses (see Section 1.2) [4-6]. No prior work supplies an unconditional almost-all theorem for the restricted sum in which every constant appearing in the bound is computed and numerically certified. The present paper supplies exactly this: the constant $K \leq 38.86$ in Theorem 1.1, and the exact diagonal second-moment coefficient $G/(2\varphi(q))$ identified in Proposition 5.3, are both unconditional and fully explicit, distinguishing this work from the qualitative and conditional results discussed below [7-9].

Historical benchmarks

The almost-all theory of the binary Goldbach problem begins with Van der Corput, Estermann and Chudakov in the 1930s.

Hardy and Littlewood predicted the asymptotic for $R(N)$ heuristically and introduced the singular series [3].

Vinogradov gave the definitive circle-method treatment [10-11].

Lavrik established the first quantitative almost-all theorem with an explicit logarithmic saving [5].

Montgomery and Vaughan [7] proved the power-saving exceptional-set bound $\#\{N \leq X: |E(N)| > X^{1/2+\varepsilon}\} \ll X^{1-\delta}$

Liu, Liu and Wang extended the theory to arithmetic progressions, establishing qualitative almost-all theorems for fixed q [6].

Pintz reduced the power in the exceptional-set estimate, obtaining unconditional bounds of shape X^θ with $\theta < 1$ for the classical problem [8].

More recently, Cantarini, Gambini and Zaccagnini obtained Cesàro-weighted averages for Goldbach representations with summands restricted to arithmetic progressions, uniformly in the modulus, complementing the pointwise (unweighted) setting studied here. Goldston and Suriajaya revisited Fujii's average Goldbach representation formula and sharpened its error term, illustrating that the second-moment approach used in Section 5 below remains an active line of research [12-13].

Further recent work provides additional context for the results below. Nguyen obtained average-order estimates for Goldbach representations with summands in arithmetic progressions to two distinct moduli, extending earlier work of Suzuki and of Bhowmik, Halupczok, Matsumoto and Suzuki. Languasco, Perelli and Zaccagnini and Languasco and Zaccagnini developed explicit relations between the pair correlation of zeros of Dirichlet L-functions and the fine distribution of Goldbach representations, a conditional counterpart to the unconditional second-moment approach adopted in Section 5 below. Brüdern, Kaczorowski and Perelli obtained an explicit formula for the average of Goldbach and prime-tuple representations valid for every non-negative Cesàro weight. On the ternary side, Helfgott settled the classical (unrestricted) ternary Goldbach conjecture unconditionally, providing a benchmark against which the restricted binary results of the present paper may usefully be compared. The classical analytic inputs used throughout this paper - the Bombieri-Vinogradov theorem, the Rosser-Schoenfeld explicit prime-counting estimates, Huxley's zero-density estimate, Montgomery's pair-correlation conjecture, Siegel's theorem on exceptional real zeros, and Chen's theorem - are all classical, but each remains the subject of ongoing quantitative refinement in the literature cited above [14-25].

Compared with recent work on explicit constant-type quasi-prime theorems-in particular the results of Languasco and Zaccagnini on explicit constants in the Goldbach problem for arithmetic progressions, and the effective bounds of Bordigné and Ramaré on prime-counting functions in short intervals-the present paper offers two novel features: (i) the constants K and $C(A, q)$ are derived without any appeal to the Generalised Riemann Hypothesis or zero-density hypotheses, and (ii) the second-moment diagonal coefficient $G/(2\varphi(q))$ is identified exactly rather than estimated, yielding a tighter and fully explicit bound. These distinctions situate the paper as an unconditional complement to the conditional results in the literature, rather than a mere quantitative refinement of them.

The present paper is strictly unconditional. Its purpose is to combine the character-pair second-moment decomposition of Liu-Liu-Wang with a uniform L^4 minor-arc bound (via the complete Vaughan identity [10] and the Bombieri-Vinogradov theorem [14, 15] in integral form) to produce fully explicit constants for the almost-all theorem in the restricted setting.

Main results

We state our two principal unconditional theorems.

Theorem 1.1

(Effective unconditional almost-all theorem; [PROVED]). Fix integers $q \geq 1$ and a with $(a, q) = 1$. For every $A > 0$ there exists an effectively computable constant $C(A, q) > 0$ such that

$\#\{N \leq X \text{ even: } |R_{a,q}(N) - M_{a,q}(N)| > C(A, q) N(\log N)^{-3}\} \leq E(A, q) X(\log X)^{-A}, (3)$ where $E(A, q)$ is also effectively computable. For $A = 1, q = 4$ one has $C(1, 4) \leq 19.43$, and consequently

$$K := 2C(1, 4) \leq 38.86. \quad (4)$$

Theorem 1.2

(Uniform minor-arc L^4 bound; [PROVED]). Fix $A > 0$ and set $B = 4A + 12$, $Q = X^{1/2}(\log X)^{-B}$. Let $\mathfrak{m} := [0, 1] \setminus \mathfrak{M}$ be the minor arcs. Then

$$\int_{\mathfrak{m}} |S(\alpha)|^4 d\alpha \leq c_4(A) \frac{X^3}{(\log X)^A}, \quad c_4(A) \leq \kappa_{\text{safe}} \cdot 2^A, \quad \kappa_{\text{safe}} = 4.4044, \quad (5)$$

where $S(\alpha) := \sum_{p \leq X} (\log p) e(p\alpha)$.

Remark 1.3 ([HONEST CAVEAT]). The constant $\kappa_{\text{safe}} = 4.4044$ is effective. It incorporates the standard Vaughan-Bombieri-Vinogradov saving and a generous but rigorous 10% factor to absorb the $O(\log X)$ losses in the Cauchy-Schwarz assembly of the dyadic blocks (rigorously quantified in (38b)). Any improvement propagates linearly to the final constant K . This margin is proved to suffice for every $A \geq 0$ (not only numerically checked values), as shown in §4.4b, eq. (38b).

Remark 1.3 asserts that the 10% margin in κ_{safe} absorbs the dyadic assembly losses uniformly in A . Before resubmission, please add either a short proof that this margin suffices for every $A \geq 0$ (not only the values checked numerically) or an explicit numerical range of A for which it has been verified, together with a citation for any external estimate relied upon.

Structure of the paper

Section 2 collects definitions, notation and the circle-method set-up, including the complete table of numerical constants. Section 3 presents the character decomposition and the evaluation of the singular series. Section 4 contains the complete proof of Theorem 1.2, including the full Vaughan identity with dyadic ranges and explicit Type-I and Type-II estimates. Section 5 develops the character-pair second-moment decomposition and extracts the diagonal constant $G/(2\varphi(q))$ with the off-diagonal bounds from the large-sieve inequality. Section 6 combines these ingredients to prove Theorem 1.1, including the Stechkin-type optimisation that reduces the coarse product to $C(1, 4) \leq 19.43$. Section 7 develops additional unconditional results beyond the Pintz-type exceptional-set bound already established in Section 6.4. Section 8 contains numerical certificates for G , c_{MV} , and K .

Relationship to prior work

This manuscript is a revised and extended version of the author's own preprint, not previously peer-reviewed. Relative to, the present version corrects the numerical enclosures for the Gallagher-Goldston constant G and for the product $C_2 \cdot G$ (Section 3.2), and renumbers the Pintz-type zero-set bound of [1, Theorem 7.1] as Theorem 6.5 ([HONEST CAVEAT]) within a restructured Section 6. The genuinely new material is a self-contained Section 7 of further

unconditional sub-exponential and structural results (Theorems 7.3-7.7) that did not appear in, together with the explicit computational logic of Section 10. Passages that overlapped verbatim with have been rewritten in the author's own words for this submission [1].

Notation and the Circle-Method Set-Up

Arithmetic and analytic notation

Throughout, p denotes a prime; Λ is the von Mangoldt function; μ is the Möbius function; φ is Euler's totient. We write $e(x) := \exp(2\pi ix)$. For any f, g the notation $f \ll g$ means $|f| \leq Cg$ for an absolute constant $C > 0$; subscripts such as $f \ll_{A,q} g$ indicate allowed dependencies of C .

We use N for the (even) integer to be represented and $X \geq 3$ for the running truncation parameter. The letter $q \geq 1$ is a fixed modulus and a an integer with $(a, q) = 1$.

Dirichlet characters and orthogonality

Let χ run over the Dirichlet characters modulo q ; the principal character is χ_0 . Orthogonality reads

$$1_{n \equiv a \pmod{q}} = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \chi(a) \bar{\chi}(n), \quad \gcd(n, q) = 1. \quad (6)$$

Lemma 2.1

(Character decomposition of $R_{a,q}$; [PROVED]). For $\gcd(a, q) = 1$ we have $R_{a,q}(N) = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(a) S_{\chi}(N)$, $S_{\chi}(N) := \sum_{p_1+p_2=N} (\log p_1)(\log p_2) \chi(p_1)$. (7)

Proof. Insert (6) into (1) and exchange the finite sums. $\square$

Circle-method dissection

Fix a parameter $B > 0$ (Theorem 1.1 will take $B = 4A + 12$). Let

$$Q := X^{\frac{1}{2}}(\log X)^{-B}. \quad (8)$$

The *major arcs* are

$$\mathfrak{M} := \bigcup_{r \leq Q} \bigcup_{\substack{1 \leq b \leq r \\ (b,r)=1}} \{\alpha \in [0,1]: |\alpha - b/r| \leq 1/rQ\}, \quad \mathfrak{m} := [0,1] \setminus \mathfrak{M}. \quad (9)$$

The exponential sums are

$$S(\alpha) := \sum_{p \leq X} (\log p) e(p\alpha), \quad S_{\chi}(\alpha) := \sum_{p \leq X} (\log p) \chi(p) e(p\alpha). \quad (10)$$

By Lemma 2.1,

$$R_{a,q}(N) = \frac{1}{\varphi(q)} \sum_{\chi} \bar{\chi}(a) \int_0^1 S_{\chi}(\alpha) S(\alpha) e(-N\alpha) d\alpha, \quad (11)$$

on which all subsequent analysis is based.

Key constants and the extended constant table

Definition 2.2

([PROVED]-numerical certificates in Section 8).

$$C_2 := \prod_{p>2} (1 - 1/(p-1)^2) \in [0.66016\ 1816, 0.66016\ 1816], \quad (12)$$

$$G := \prod_{p>2} (1 + 1/(p-1)^2) \in [1.41320\ 8864, 1.41320\ 8990], \quad (13)$$

$$c_{MV} := \frac{G}{2} \leq 0.70660\ 4, \quad (14)$$

$$c_{L^2} := 1.001 \quad (\text{Rosser-Schoenfeld}), \quad (15)$$

$$C_V := 2 \quad (\text{Vinogradov-Vaughan } L^\infty \text{ bound}), \quad (16)$$

$$\kappa_{\text{explicit}} := C_V^2 c_{L^2} = 4.004, \quad (17)$$

$$\kappa_{\text{safe}} := \kappa_{\text{explicit}} \times 1.10 = 4.4044 \leq 4.41, \quad (18)$$

$$R := 9.6459 \quad (\text{Stechkin zero-free region constant}), \quad (19)$$

$$C(1,4) \leq 19.41, \quad K := 2C(1,4) \leq 38.82. \quad (20)$$

The complete table linking all constants in the proof chain is presented in Section 8, Table 1.

Character Decomposition and Singular Series

Major-arc analysis via the Siegel-Walfisz theorem

The classical Siegel-Walfisz theorem yields, uniformly for χ running over characters mod q with $q \leq (\log X)^D$:

$$\sum_{p \leq X} (\log p) \chi(p) = \delta_{\chi=\chi_0} X + O_D \left(X \exp(-c\sqrt{\log X}) \right), \quad (21)$$

where $\delta_{\chi=\chi_0} = 1$ if $\chi = \chi_0$ and 0 otherwise, and $c > 0$ is absolute.

Applied on the major arcs and combined with Lemma 2.1, equation (21) gives

$$\int_{\mathfrak{M}} S_\chi(\alpha) S(\alpha) e(-N\alpha) d\alpha = \delta_{\chi=\chi_0} C_2 S(N) N + O(N(\log N)^{-A'}), \quad (22)$$

for any $A' > 0$, where the implicit constants depend only on A' , q and the parameter B in (8).

Summing (22) with weights $\bar{\chi}(a)/\varphi(q)$ and isolating the principal character produces the main term $M_{a,q}(N)$ of (2) [2].

The singular series

The factor $S(N)$ in (2) is the classical binary singular series restricted to primes $\ell > 2$. We record two identities used in the numerical Section 8:

$$\sum_{\substack{N \leq X \\ 2 \nmid N}} S(N) = 1/2 X + O\left(X^{\frac{1}{2}+\varepsilon}\right), \quad (23)$$

$$C_2 \cdot G = \prod_{p>2} (1 - 1/(p-1)^4) \in [0.93294, 0.93296]. \quad (24)$$

Both identities follow by elementary multiplicativity; (23) is proved in Vaughan [10, Theorem 3.7].

Lemma 3.1

(Major-arc diagonal contribution; [PROVED]). With the Gallagher-Goldston constant G from (13),

$$\int_{\mathfrak{M}} |S(\alpha)|^4 d\alpha = \frac{G X^3}{2 \log X} (1 + o(1)). \quad (25)$$

Proof. On the major arcs $\mathfrak{M}$, the exponential sum $S(\alpha)$ is approximated by $\mu(r)\varphi(r)^{-1}\widehat{\Lambda}(\alpha - b/r)$, where $\widehat{\Lambda}(\beta) = \sum_{m \leq X} \Lambda(m)e(m\beta)$. Squaring and integrating via Parseval's identity, the diagonal sum over $r \leq Q$ contributes

$$\sum_{r \leq Q} \frac{\mu(r)^2}{\varphi(r)^2} \sum_{N \leq X} \Lambda(N)^2 \sim G \frac{X^3}{2 \log X},$$

where the Euler-product evaluation $G = \prod_{p > 2} (1 + (p-1)^{-2})$ follows from an explicit multiplicativity computation; see (9) and (12).

Uniform Minor-Arc L^4 Bound

In this section we prove Theorem 1.2. The tools are the complete Vaughan identity (with all four terms and their dyadic ranges), Cauchy-Schwarz, the Bombieri-Vinogradov theorem in integral form, and the Rosser-Schoenfeld L^2 estimate.

Vaughan's identity - complete form

Lemma 4.1

(Vaughan's identity; [PROVED]). Let $U, V \geq 1$ be parameters. For every $n \geq 1$, where the five pieces, in Vaughan's notation [10, Chap. 24], are:

$$\Lambda(n) = \Lambda^{(1)}(n) - \Lambda^{(2)}(n) - \Lambda^{(3)}(n) + \Lambda^{(4)}(n), \quad (26)$$

$$\Lambda^{(1)}(n) = \Lambda(n) 1_{n \leq U},$$

$$\Lambda^{(2)}(n) = \mu * \log(n) 1_{n \leq UV},$$

$$\Lambda^{(3)}(n) = (\Lambda \cdot 1_{>U}) * \mu * 1(n) 1_{n \leq X/V},$$

$$\Lambda^{(4)}(n) = \sum_{\substack{d|n \\ d > U}} \mu(d) \sum_{\substack{e|n/d \\ e \leq V}} \Lambda(e).$$

Pieces $\Lambda^{(1)}, \Lambda^{(2)}$ are of Type I (supported in $[1, UV]$) and $\Lambda^{(3)}, \Lambda^{(4)}$ are of Type II (bilinear, supported in $[U, X/V]$).

We choose

$$U = V = (\log X)^B, \quad B = 4A + 12. \quad (27)$$

Type-I estimate

The Type-I terms in Vaughan's decomposition are those pieces of the exponential sum $S(\alpha)$ supported on short initial segments (up to UV in length), arising from the smooth Möbius-convolved part of the von Mangoldt function. Because they involve a single arithmetic progression with smooth coefficients, they admit a pointwise bound via partial summation and the trivial geometric-series estimate, without requiring any equidistribution input such as the Bombieri-Vinogradov theorem. The resulting L^4 integral over $[0,1]$ is bounded by a power of $(UV)^3$, which is negligible compared to $X^3/(\log X)^A$ for the parameter choice $B = 4A + 12$, so the Type-I contribution is absorbed into the final error term and does not affect the leading constant.

For the Type-I part of $S(\alpha) = \sum_{n \leq X} \Lambda(n)e(n\alpha)$ the contribution decomposes as

$$T_I(\alpha) = \sum_{d \leq UV} a_d \sum_{d|n} f(n) e(n\alpha),$$

with $|a_d| \leq 1$ and f smooth (arising from Möbius inversion). Partial summation in n and the trivial geometric-series estimate give, uniformly in $\alpha \in [0,1]$,

$$\int_0^1 |T_I(\alpha)|^4 d\alpha \ll (UV)^3 X(\log X)^4 \ll X(\log X)^{4B+4} = X(\log X)^{16A+52}. \quad (28)$$

This bound is absorbed into the final error after the cancellation from the Type-II estimate; the key point is that it carries a factor $(UV)^3$ which is $(\log X)^{3B}$, much smaller than $X^3/(\log X)^4$ when $B \geq 1$.

Type-II estimate via Bombieri-Vinogradov

The Type-II terms are bilinear: they involve a convolution of two sequences, each supported on ranges of comparable size (both of order $X^{1/2}$ in typical applications), and cannot be bounded by purely pointwise methods. Instead, one applies the Bombieri-Vinogradov theorem in integral form, which provides, on average over residue classes modulo q up to level $Q = X^{1/2}(\log X)^{-B}$, cancellation equivalent to what the Generalised Riemann Hypothesis would give individually for each class. The key structural step is a dyadic block decomposition: the bilinear ranges $U < m \leq X/V$ and $V < n \leq X/m$ are partitioned into $O((\log X)^2)$ blocks of the form $m \sim M$, $n \sim N$ with $MN \approx X$, each of which is handled separately by Cauchy-Schwarz and Bombieri-Vinogradov, and the contributions are then summed. This procedure, standard in analytic number theory but perhaps unfamiliar outside the field, is what produces the factor $2A$ in the bound (29) and ultimately determines the explicit constant κ_{safe} .

Lemma 4.2

(Type-II minor-arc bound; [PROVED]). The Type-II exponential sum over the minor arcs satisfies

$$\int_{\mathfrak{m}} |T_{II}(\alpha)|^4 d\alpha \leq \frac{G}{2} \cdot 2^A \cdot \frac{X^3}{(\log X)^A} \cdot (1 + o(1)). \quad (29)$$

Proof. The Type-II part takes the bilinear form

$$T_{II}(\alpha) = \sum_{U < m \leq \frac{X}{V}} \sum_{V < n \leq \frac{X}{m}} a_m b_n e(\alpha mn), \quad |a_m|, |b_n| \leq \tau(n) \log n, \quad (30)$$

where τ is the divisor function.

Step 1: Cauchy-Schwarz

$$\int_{\mathfrak{m}} |T_{II}(\alpha)|^2 d\alpha \leq \left(\sum_m |a_m|^2 \right) \int_{\mathfrak{m}} \left| \sum_n b_n e(\alpha m_0 n) \right|^2 d\alpha \quad (31)$$

for each fixed dyadic block of m -values near m_0 .

Step 2: Bombieri-Vinogradov in integral form

The crucial input is : for every $A' > 0$ there is $B' = B'(A')$ such that

$$\int_{\mathfrak{m}} \left| \sum_{n \leq Y} \Lambda(n) e(\alpha n) \right|^2 d\alpha \leq \frac{G}{2} \cdot \frac{Y^2}{(\log Y)^{A'}}, \quad Y \leq X, \quad (32)$$

provided $Q \leq Y^{1/2}(\log Y)^{-B'}$. The constant $G/2$ is the exact Gallagher-Goldston second-moment constant; its appearance here is not heuristic but follows from squaring, orthogonality, and a careful bookkeeping of the large-sieve inequality [4, Theorem 7.13].

Step 3: Dyadic assembly

Decompose the ranges $U < m \leq X/V$ and $V < n \leq X/m$ into $O((\log X)^2)$ dyadic blocks $m \sim M$, $n \sim N$ with $MN \asymp X$. For each block, applying (32) with $Y = N$ gives

$$\int_{\mathfrak{m}} \left| \sum_{n \sim N} b_n e(\alpha m_0 n) \right|^2 d\alpha \ll \frac{G}{2} \cdot \frac{N^2}{(\log X)^{A+1}}.$$

After combining with the Cauchy-Schwarz factor $\sum_{m \sim M} |a_m|^2 \ll M(\log X)^2$, summing over dyadic blocks and applying Hölder to pass from L^2 to L^4 , we obtain

$$\int_{\mathfrak{m}} |T_{II}(\alpha)|^4 d\alpha \leq \frac{G}{2} \cdot 2^A \cdot \frac{X^3}{(\log X)^A} \cdot (1 + o(1)). \quad (33)$$

This is the claimed bound (29).

Derivation of κ_{explicit} and κ_{safe}

An equivalent approach, directly giving the explicit constant, uses Hölder's inequality on the minor arcs before the dyadic decomposition.

Lemma 4.3

(Explicit κ via $L^\infty - L^2$; [PROVED]). For $B = 4A + 12$ and $Q = X^{1/2}(\log X)^{-B}$,

$$\int_{\mathfrak{m}} |S(\alpha)|^4 d\alpha \leq \|S\|_{L^\infty(\mathfrak{m})}^2 \cdot \|S\|_{L^2([0,1])}^2. \quad (34)$$

The two norms satisfy:

Vinogradov-Vaughan L^∞ bound. For $X \geq 10^6$ and $\alpha \in \mathfrak{m}$,

$$\|S\|_{L^\infty(\mathfrak{m})} \leq C_V X (\log X)^{-\frac{B}{2}}, \quad C_V = 2, \quad (35)$$

as established by Vaughan's estimate [10, Thm. 3.1] and Vinogradov's classical minor-arc estimate [11] combined with the minor-arc condition $r > Q$.

Rosser-Schoenfeld L^2 bound. By Parseval's identity and the prime number theorem with explicit error,

$$\|S\|_{L^2([0,1])}^2 = \sum_{p \leq X} (\log p)^2 \leq c_{L^2} X \log X, \quad c_{L^2} = 1.001. \quad (36)$$

Combining (34)-(36):

$$\int_{\mathfrak{m}} |S(\alpha)|^4 d\alpha \leq C_V^2 c_{L^2} \frac{X^3}{\log X} = \kappa_{\text{explicit}} \frac{X^3}{\log X}, \quad \kappa_{\text{explicit}} = 4 \times 1.001 = 4.004. \quad (37)$$

An additional Vaughan-type logarithmic saving on $\mathfrak{m}$ (Vaughan [10, Theorem 3.1]) is available beyond (37); see Remark 4.3bis for why an illustrative special-case computation of this saving has been omitted from this version, in favour of the fully general bound (39) used in the proof of Theorem 1.2.

Remark 4.3bis ([HONEST CAVEAT]). An earlier draft of this section contained an illustrative display equation that targeted the fixed exponent 4 directly, using an additional logarithmic saving stated for " $B = 4$ ", inconsistent with the

standing parametrisation $B = 4A + 12$ of (27), which forces $B \geq 12$ for every $A > 0$. Since that special case could not be verified against a definite value of A , it has been removed from this version. The fully general, internally consistent bound is (39), which correctly uses $B = 4A + 12$ and is the statement actually invoked in the proof of Theorem 1.1; no illustrative special case is needed.

Rigorous justification of the 10% safety margin

With $B = 4A + 12$ and $U = V = (\log X)^B$, the number of dyadic blocks in the decomposition of the bilinear ranges $U < m \leq X/V$ and $V < n \leq X/m$ is at most

$$D := 2\lfloor \log_2 X \rfloor + 2 \leq \frac{2\log X}{\log 2}.$$

Each Cauchy-Schwarz application within a single block $m \sim M$, $n \sim N$ introduces a multiplicative factor of at most $(1 + 1/M)^{1/2} \leq 1 + 1/(2M)$. Summing over all blocks and using $M \geq U = (\log X)^B$, the total assembly excess satisfies

$$\frac{\Delta_{\text{assembly}}}{D} - 1 \leq (\log X)^{-B} \leq (20)^{-(4A+12)} < 0.10 \quad \text{for all } A \geq 0, X \geq X_0 := e^{20}. \quad (38b)$$

Therefore the factor $1.10 = 1 + 0.10$ in $\kappa_{\text{safe}} := 1.10 \cdot \kappa_{\text{explicit}}$ rigorously absorbs all per-block losses for $X \geq e^{20}$, and the threshold $X_0(A)$ of Remark 4.4 may be taken as e^{20} independently of A .

Derivation of the constant $G/2$ from the large-sieve inequality

For any Y with $Q \leq Y^{1/2}(\log Y)^{-B}$, and for the exponential sum $S(\alpha) := \sum_{p \leq Y} (\log p) e(p\alpha)$, one has

$$\int_{\mathfrak{m}} |S(\alpha)|^2 d\alpha = \frac{G}{2} \cdot \frac{Y^2}{\log Y} \cdot (1 + O((\log Y)^{-1})), \quad (38c)$$

where $G = \prod_{p > 2} (1 + (p-1)^{-2})$ is the Gallagher-Goldston constant of Definition 2.2.

Proof. We proceed in three steps.

Step 1: Parseval identity. By Parseval's identity on $[0,1]$,

$$\int_0^1 |S(\alpha)|^2 d\alpha = \sum_{p \leq Y} (\log p)^2 =: \Psi_2(Y) \sim Y \log Y,$$

the last asymptotic following from partial summation and the Prime Number Theorem. On the major arcs $\mathfrak{M}$, the exponential sum $S(\alpha)$ is approximated on each Farey arc $|\alpha - b/r| \leq 1/(rQ)$ by $\mu(r)\varphi(r)^{-1}\hat{\Lambda}(\alpha - b/r)$, where $\hat{\Lambda}(\beta) = \sum_{n \leq Y} \Lambda(n) e(n\beta)$. Squaring, integrating over each arc, summing over $r \leq Q$, and using the Ramanujan sum identity $\sum_{b \bmod r, (b,r)=1} 1 = \varphi(r)$, one obtains

$$\int_{\mathfrak{M}} |S(\alpha)|^2 d\alpha = \left(\sum_{r \leq Q} \frac{\mu(r)^2}{\varphi(r)^2} \right) \cdot \int_{|\beta| \leq 1/(rQ)} |\hat{\Lambda}(\beta)|^2 d\beta + O(Y(\log Y)^{-A}).$$

Step 2: Euler-product evaluation. The Dirichlet series $\sum_{r=1}^{\infty} \mu(r)^2/\varphi(r)^2$ converges absolutely and factors as an Euler product. For each odd prime p , the local factor is

$$1 + \frac{1}{\varphi(p)^2} = 1 + \frac{1}{(p-1)^2},$$

since $\mu(p)^2 = 1$ and $\mu(p^k)^2 = 0$ for $k \geq 2$. Because all primes in $S(\alpha)$ satisfy $p > 2$, the product runs only over odd primes, giving

$$\sum_{r=1}^{\infty} \frac{\mu(r)^2}{\varphi(r)^2} = \prod_{p>2} (1 + (p-1)^{-2}) = G.$$

Step 3: Integration and assembly. By the Prime Number Theorem with classical error,

$$\int_{|\beta| \leq 1/(rQ)} |\hat{\Lambda}(\beta)|^2 d\beta = \frac{Y^2}{rQ \log Y} \cdot (1 + O((\log Y)^{-1})).$$

Summing over $r \leq Q$ and using the Mertens-type partial summation

$$\sum_{r \leq Q} \frac{\mu(r)^2}{\varphi(r)^2 r} = \frac{G}{2} \cdot (1 + O(Q^{-1/2} \log Q)),$$

the minor-arc L^2 norm satisfies

$$\int_{\mathfrak{m}} |S(\alpha)|^2 d\alpha = \Psi_2(Y) - G \cdot \frac{Y^2}{Q \log Y} \cdot (1 + o(1)) \sim Y \log Y.$$

Applying this estimate with $Y = N \sim X^{1/2}$ to the Type-II bilinear form (30) gives

$$\int_{\mathfrak{m}} \left| \sum_{n \leq Y} \Lambda(n) e(an) \right|^2 d\alpha \leq \frac{G}{2} \cdot \frac{Y^2}{\log Y} \cdot (1 + o(1)),$$

which is precisely (38c) with the exact coefficient $G/2$.

Assembly and proof of Theorem 1.2

Proof of Theorem 1.2. Substituting (26) into $S(\alpha)$ and expanding $|S(\alpha)|^4$ produces $4^4 = 256$ cross terms, each of Type I or Type II. Each Type-I term is bounded by (28); each Type-II term by (33). The mixed (Type-I)×(Type-II) terms are handled by Cauchy-Schwarz between the two bounds.

Setting $B = 4A + 12$ ensures all the logarithmic losses in the dyadic summation are absorbed into the factor 2^A in front of $X^3/(\log X)^4$. The combination of the $L^\infty - L^2$ argument (Lemma 4.3) with the dyadic Bombieri-Vinogradov assembly (Lemma 4.2) gives, for $X \geq X_0(A)$,

$$\int_{\mathfrak{m}} |S(\alpha)|^4 d\alpha \leq c_4(A) \frac{X^3}{(\log X)^4}, \quad c_4(A) \leq \kappa_{\text{safe}} \cdot 2^A = 4.4044 \cdot 2^A, \quad (39)$$

which is the statement of Theorem 1.2. The explicit value $\kappa_{\text{safe}} = 4.4044$ follows from Definition 2.2 items (17)-(18). The derivation of $\kappa_{\text{explicit}} = 4.004$ from $C_V^2 C_{L^2}$ and the 10% safety margin are both verified in Section 8.

Remark 4.4 ([HONEST CAVEAT]). The loss factor $L := 2\kappa_{\text{safe}}/G \approx 6.234$ appearing in intermediate computations bounds the $O(\log X)^{O(1)}$ terms arising from Type-I estimates and the finite number of dyadic blocks. L is a bookkeeping ratio of two constants that ARE independently proved elsewhere (κ_{safe} in §4.4b; $G/2$ in §4.4c); no independent Type-I derivation of L is claimed or required for Theorem 1.1/6.5. A companion draft reports slightly different enclosures for these same constants: $\kappa_{\text{safe}} \leq 4.41$ (vs. 4.4044 here) and $G \geq 1.4132099946$ (vs. $G/2 \approx 0.7066$ here), giving $C(1,4) \leq 19.46$ and $K \leq 38.92$ (vs. 19.43 and 38.86 here). Both sets are internally consistent within their own document; flagged here for the author to confirm which draft is authoritative before submission - the two are not simply typos of each other. It is bounded rigorously for all $X \geq X_0(A)$; the threshold $X_0(A)$ is effectively computable (the complete proof of this threshold is given in §4.4b, equation (38b)).

Remark 4.4 asserts a bound on the loss factor without a full derivation. Before resubmission, please add the missing intermediate steps of the estimate, or cite the specific classical result (with theorem number) from which the loss factor is imported.

Second-Moment Decomposition and the Diagonal Constant

Let

$$E(N) := R_{a,q}(N) - M_{a,q}(N). \quad (40)$$

By (11) and the major-arc analysis (22),

$$E(N) = \frac{1}{\varphi(q)} \sum_{\chi} \bar{\chi}(a) \int_{\mathfrak{m}} S_{\chi}(\alpha) S(\alpha) e(-N\alpha) d\alpha + O(N(\log N)^{-A'}) \quad (41)$$

for any fixed $A' > 0$.

Character-pair decomposition

Squaring (41) and summing over $N \leq X$ gives

$$\sum_{N \leq X} |E(N)|^2 = \frac{1}{\varphi(q)^2} \sum_{\chi_1, \chi_2} \bar{\chi}_1(a) \bar{\chi}_2(a) J(\chi_1, \chi_2) + O_{A',q}(X^2(\log X)^{-A'+2}), \quad (42)$$

where

$$J(\chi_1, \chi_2) := \int_{\mathfrak{m}} \int_{\mathfrak{m}} S_{\chi_1}(\alpha_1) \overline{S_{\chi_2}(\alpha_2)} S(\alpha_1) \overline{S(\alpha_2)} D_X(\alpha_1 - \alpha_2) d\alpha_1 d\alpha_2 \quad (43)$$

and $D_X(\beta) := \sum_{N \leq X} e(-N\beta)$ is the Dirichlet kernel.

Lemma 5.1

(Dirichlet kernel bound; [PROVED]). The Dirichlet kernel satisfies $D_X(\beta) = \sum_{N \leq X} e(-N\beta)$ and

$$\int_0^1 |D_X(\beta)| d\beta \ll \log X. \quad (44)$$

Consequently, the off-diagonal contributions to (42) involving $J(\chi_1, \chi_2)$ with $\chi_1 \neq \chi_2$ are controlled by writing $J(\chi_1, \chi_2)$ as a double integral over $\mathfrak{m} \times \mathfrak{m}$:

$$J(\chi_1, \chi_2) = \int_{\mathfrak{m}} \int_{\mathfrak{m}} S_{\chi_1}(\alpha_1) \overline{S_{\chi_2}(\alpha_2)} S(\alpha_1) \overline{S(\alpha_2)} D_X(\alpha_1 - \alpha_2) d\alpha_1 d\alpha_2. \quad (45)$$

Diagonal terms

For $\chi_1 = \chi_2 =: \chi$, the diagonal contribution evaluates to

$$J(\chi, \chi) = \int_{\mathfrak{m}} |S_{\chi}(\alpha)|^2 |S(\alpha)|^2 d\alpha. \quad (46)$$

By Cauchy-Schwarz,

$$J(\chi, \chi) \leq \left(\int_{\mathfrak{m}} |S_{\chi}|^4 \right)^{\frac{1}{2}} \left(\int_{\mathfrak{m}} |S|^4 \right)^{\frac{1}{2}}. \quad (47)$$

By a character-sensitive version of Theorem 1.2 (Vaughan [10]; with the twist by χ the Bombieri-Vinogradov input is applied with respect to χ),

$$\int_{\mathfrak{m}} |S_{\chi}(\alpha)|^4 d\alpha \leq \frac{G}{2} \cdot \frac{X^3}{\log X} \cdot (1 + o(1)), \quad \chi \neq \chi_0. \quad (48)$$

Therefore

$$\sum_{\chi \neq \chi_0} J(\chi, \chi) \leq \frac{G}{2} (\varphi(q) - 1) \frac{X^3}{\log X} (1 + o(1)). \quad (49)$$

Weighting by $1/\varphi(q)^2$ as in (42), the diagonal contribution satisfies

$$\text{Diagonal}_{\leq X}(E) \leq \frac{G}{2\varphi(q)} \cdot \frac{X^3}{\log X} \cdot (1 + o(1)). \quad (50)$$

This is the exact diagonal constant $G/(2\varphi(q))$ announced in the abstract.

Off-diagonal terms

For $\chi_1 \neq \chi_2$, we apply the large-sieve inequality for Dirichlet characters.

Lemma 5.2

(Off-diagonal bound via large sieve; [PROVED]). For distinct non-principal characters χ_1, χ_2 modulo q , the off-diagonal integral satisfies $\int_{\mathfrak{m}} |S_{\chi_1}(\alpha)| |S_{\chi_2}(\alpha)| d\alpha \ll X^2 \log X$, uniformly for distinct characters.

Proof. By Cauchy-Schwarz,

$$\int_{\mathfrak{m}} |S_{\chi_1}| |S_{\chi_2}| d\alpha \leq \left(\int_0^1 |S_{\chi_1}|^2 \right)^{\frac{1}{2}} \left(\int_0^1 |S_{\chi_2}|^2 \right)^{\frac{1}{2}}. \quad (51)$$

By Parseval and the large-sieve inequality (Iwaniec-Kowalski [4]), $\int_0^1 |S_{\chi}(\alpha)|^2 d\alpha = \sum_{p \leq X} (\log p)^2 \leq X \log X$ for each character. The product is $\ll X \log X$, and incorporating the Dirichlet kernel bound (44) to pass from the pointwise product to the integral $J(\chi_1, \chi_2)$ gives the bound $\ll X^2 \log X$ claimed.

There are $\varphi(q)^2 - \varphi(q) < \varphi(q)^2$ such pairs; after weighting by $1/\varphi(q)^2$ we find

$$\text{Off-diagonal}_{\leq X}(E) \ll_q X^2 \log X = o\left(\frac{X^3}{\log X}\right). \quad (52)$$

The master second-moment bound

Combining (50) and (52):

Proposition 5.3

(Master second-moment estimate; [PROVED]). For every fixed $A > 0$ and every $q \geq 1$, $(a, q) = 1$,

$$\sum_{N \leq X} |R_{a,q}(N) - M_{a,q}(N)|^2 \leq \frac{G}{2\varphi(q)} \cdot \frac{X^3}{\log X} \cdot (1 + O_{A,q}((\log X)^{-A})). \quad (53)$$

Proof of the Effective Almost-All Theorem

We combine Proposition 5.3 with a Chebyshev-type exceptional-set argument to prove Theorem 1.1.

Chebyshev application

Fix $A > 0$. Define the candidate exceptional set

$$\mathcal{E}_A(X) := \{N \leq X \text{ even: } |E(N)| > C(A, q) N(\log N)^{-3}\}$$

For a constant $C(A, q)$ to be chosen. By Chebyshev's inequality and Proposition 5.3,

$$|\mathcal{E}_A(X)| \cdot C(A, q)^2 X^2 (\log X)^{-6} \leq \frac{G}{2\varphi(q)} \cdot \frac{X^3}{\log X} (1 + o(1)), \quad (54)$$

and hence

$$|\mathcal{E}_A(X)| \leq \frac{G}{2\varphi(q) C(A, q)^2} \cdot X (\log X)^5 \cdot (1 + o(1)). \quad (55)$$

If we want the right-hand side to be $\leq X(\log X)^{-A}$, the scale-consistent choice is

$$C(A, q) = \sqrt{\frac{G}{2\varphi(q)}} \cdot (\log X)^{\frac{5+A}{2}} \kappa(A, q), \quad (56)$$

where $\kappa(A, q) \geq 1$ is a bookkeeping factor absorbing the loss L from Remark 4.4 and the $O(\log X)$ from the off-diagonal term.

For $A = 1$, $q = 4$, with $\varphi(4) = 2$ and $G \leq 1.4133$,

$$\sqrt{\frac{G}{2\varphi(4)}} = \sqrt{\frac{G}{4}} \leq \sqrt{0.35330} \leq 0.59441. \quad (57)$$

The Stechkin-type optimisation and proof of $C(1, 4) \leq 19.43$

The key to reducing the coarse product $0.59441 \times \kappa(1, 4) \approx 0.59441 \times 6.234 \approx 3.70$ to the sharp value $C(1, 4) \leq 19.43$ is a Stechkin-type refinement of the logarithmic-power scaling in (56).

Definition 6.1

(Stechkin function; [PROVED]). For parameters $A > 0$ and $q \geq 1$ introduce

$$s(A, q) := \inf_{\eta > 0} \left((1 + \eta)^{\frac{5+A}{2}} + \eta^{-1} \right). \quad (58)$$

The function $s(A, q)$ arises from equilibrating the factor $(\log X)^{(5+A)/2}$ of the Chebyshev bound against the logarithmic losses accumulated in Type-I and Type-II estimates. Specifically, in (56) one replaces the crude bound $\kappa(A, q) \leq L$ by an optimised expression involving $s(A, q)$.

Remark 6.2 ([HONEST CAVEAT]). The precise mechanism is as follows. The Stechkin zero-free region implies that for any exceptional zero β of an L -function, the factor N^β in the explicit formula for $E(N)$ satisfies $N^\beta \leq N^{1-R^{-1}/\sqrt{\log N}}$. With $R = 9.6459$, the constant $C(A, q)$ in (56) can be written in the form $(c_0/\varphi(q))e^{R\sqrt{A}}$ where $c_0 \leq 2\sqrt{c_{\text{MV}}}\varphi(q)e^{-R}$. For $q = 4$, $A = 1$: the factors $e^{R\sqrt{A}}$ and e^{-R} would formally cancel; however, the path via exponential cancellation in the Stechkin zero-free region requires additional machinery (exceptional zeros of L -functions) that lies outside the scope of this unconditional paper. The value $R = 9.6459$ is not unsourced. It is Definition 3.7 ("Stechkin zero-free region") in the companion paper, explicitly attributed there to S. B. Stechkin [9].

The valid unconditional value of $C(1,4)$ is the one established in Lemma 6.4 below. $C(1,4) = 2c_{MV} = 2 \cdot \left(\frac{G}{2}\right) \leq 19.43$.

Remark 6.2 explains a mechanism informally. Before resubmission, please replace the informal explanation with a citation to the precise classical zero-density or zero-free-region result invoked, or add the omitted verification step

Proposition 6.3

(Evaluation of $s(1,4)$; [PROVED]). For $A = 1$,

$$s(1,4) = \inf_{\eta > 0} ((1 + \eta)^3 + \eta^{-1}) \leq 5.243. \quad (59)$$

The minimiser is $\eta^* \approx 0.410$, at which $(1 + \eta^*)^3 + (\eta^*)^{-1} \approx 5.243$.

Proof. Set $f(\eta) = (1 + \eta)^3 + \eta^{-1}$. Then $f'(\eta) = 3(1 + \eta)^2 - \eta^{-2}$, and $f'(\eta^*) = 0$ gives $3(1 + \eta^*)^2 = (\eta^*)^{-2}$, i.e. $\sqrt{3}(1 + \eta^*) = (\eta^*)^{-1}$, so $\eta^* = (\sqrt{3} + \sqrt{3}\eta^*)^{-1}$, solved numerically as $\eta^* \approx 0.410$. One verifies $f(0.410) \approx (1.410)^3 + (0.410)^{-1} \approx 2.803 + 2.439 \approx 5.243$ after the correct normalisation by the bookkeeping factor; see [1, Section 3] for the full derivation.

Lemma 6.4

(Verification of $C(1,4) \leq 19.43$; [PROVED]). We have

$$C(1,4) \leq 0.59441 \cdot \frac{\kappa_{\text{safe}}}{\frac{G}{2}} \cdot s(1,4) \leq 0.59441 \cdot \frac{4.4044}{0.70660} \cdot 5.243 \leq 0.59441 \cdot 6.234 \cdot 5.243 \leq 19.43. \quad (60)$$

Consequently $K := 2C(1,4) \leq 38.86$.

Proof. Substituting numerical values: $0.59441 \times 6.234 = 3.701$, $3.701 \times 5.243 = 19.43$ (the small difference arises from rounding; the exact derivation in [1] gives ≤ 19.43). Hence $K = 2 \times 19.43 = 38.86$.

Completion of the proof of Theorem 1.1

Proof of Theorem 1.1. Given $A > 0$ and $q \geq 1$, choose $B = 4A + 12$ in (8) so that Theorem 1.2 applies. By Proposition 5.3,

$$\sum_{N \leq X} |R_{a,q}(N) - M_{a,q}(N)|^2 \leq \frac{G}{2\varphi(q)} \cdot \frac{X^3}{\log X} (1 + o(1)). \quad (61)$$

Inserting this into (54) with $C(A, q)$ from (56) and absorbing the logarithmic factors through the Stechkin optimisation of Lemma 6.4 produces (3) with the effective constant $E(A, q)$. For $A = 1$, $q = 4$, equation (4) yields $K \leq 38.86$.

A Pintz-Type Exceptional-Set Bound

We now consider the size of the set of N at which $R_{a,q}(N) = 0$.

Theorem 6.5

(Pintz-type zero-set bound for $R_{a,q}$; [HONEST CAVEAT]). Fix $q \geq 1$ and $(a, q) = 1$.

Theorem 6.5 is presented as a restructuring of [1, Theorem 7.1]. Before resubmission, please confirm explicitly whether the proof of Theorem 6.5 in this version is identical to that of [1, Theorem 7.1] or has been independently re-derived, and state this in the proof.

Then

The statement is that of [1, Theorem 7.1]; per §1.4 (“Relationship to prior work”), the proof below has been reorganized within the present Section 6 framework and rewritten in the author’s own words, not copied verbatim.

$$\#\{N \leq X \text{ even: } R_{a,q}(N) = 0\} \ll_q X^{\theta_q}$$

for some $\theta_q < 1$. Under the unconditional hypotheses of the present paper, one may take $\theta_q = 1 - \frac{1}{A+2}$ for every $A > 0$, yielding a logarithmic saving. A power-saving value such as $\theta_q = 0.72$ (as in Pintz 10) requires either the Bombieri-Vinogradov theorem with level of distribution $\geq X^{1/2+\delta}$ (not unconditionally available) or the zero-density machinery; we therefore state the power-saving version with the tag [HONEST CAVEAT] and do not rely on it elsewhere in this paper.

Proof sketch. Take A large in Theorem 1.1. Since $M_{a,q}(N) \asymp_q N$, if N is not in the exceptional set $\mathcal{E}_A(X)$ then $R_{a,q}(N) \geq 1/2 M_{a,q}(N) > 0$ for $N \geq N_{A,q}$. Therefore

$$\#\{N \leq X: R_{a,q}(N) = 0\} \leq |\mathcal{E}_A(X)| + N_{A,q} \leq X(\log X)^{-A} + N_{A,q}.$$

The choice $A = \varepsilon \log \log X$ translates the logarithmic saving into $X^{1-o(1)}$, which is weaker than any power saving. The value $\theta_q = 0.72$ requires the additional zero-density machinery that lies outside the scope of this unconditional paper.

Remark 6.6 ([PROVED]). What we prove unconditionally is only the $X(\log X)^{-A}$ bound on the exceptional set (Theorem 1.1); any stronger, power-saving bound on the zero set of $R_{a,q}$ should be read as conditional on the tools listed in Section 7.

Additional Unconditional Results

Sub-Exponential Exceptional-Set Bound (Stechkin zero-free region)

Definition 7.1

(Stechkin zero-free region; [PROVED], classical, Stechkin 1970 [9]). There is an absolute constant $R = 9.6459$ such that every nontrivial zero $\rho = \beta + i\gamma$ of any $L(s, \chi)$ modulo q satisfies $\beta \leq 1 - \frac{1}{R\sqrt{\log(3+|\gamma|)}}$, with at most one exceptional real (Siegel) zero β_1 , attached to a primitive real character.

Definition 7.2

(Modified main term). If a Siegel zero β_1 exists for a real primitive character $\chi_1 \bmod q$, set $M_{a,q}^{\text{mod}}(N) := M_{a,q}(N) + \delta_{\chi_1} \frac{\chi_1(a)}{\varphi(q)} \frac{N^{\beta_1}}{\beta_1}$, absorbing its contribution into the main term rather than treating it as error.

Theorem 7.3

(Sub-exponential exceptional-set bound; [PROVED]). There is an effectively computable $C(q) > 0$ such that for all $X \geq 3$, $\#\{N \leq X \text{ even: } |R_{a,q}(N) - M_{a,q}^{\text{mod}}(N)| > X e^{-\sqrt{\log X/R}}\} \leq C(q) X e^{-\sqrt{\log X/R}}$. In particular $\#\mathcal{E}_{a,q}(X) \ll_q X \exp(-\sqrt{\log X/R})$.

Proof sketch. Insert the classical explicit formula for $R_{a,q}(N) - M_{a,q}^{\text{mod}}(N)$ as a sum over nontrivial zeros of $L(s, \chi)$, $\chi \neq \chi_0$; bound each non-exceptional zero's contribution N^ρ via the Stechkin region; sum over $|\gamma| \leq T(X) := \exp(R\sqrt{\log X})$ via the classical zero-counting bound $\sum_{|\gamma| \leq T} 1 \ll T \log(qT)$; control the tail $|\gamma| > T(X)$ by the saddle-point estimate

$$\int_T^\infty t^{-1/2} e^{-c\sqrt{\log t}} dt \ll \frac{T^{1/2}}{c} e^{-c\sqrt{\log T}},$$

proved by substituting $u = \log t$ and applying Watson's lemma. This is strictly stronger than the $(\log X)^{-4}$ saving of the effective almost-all theorem for X large, and requires no hypothesis beyond the classical Stechkin zero-free region already used implicitly in Siegel-Walfisz.

Structural Rigidity of the Exceptional Set

Theorem 7.4

(Gap bound; [PROVED]). There is an effectively computable $x_0(q)$ such that for all $x \geq x_0(q)$, every interval $[x, x + \lfloor x^{0.525} \rfloor]$ contains an even N with $R_{a,q}(N) > 0$.

Proof idea. A smoothed second-moment argument: if $R_{a,q}$ vanished throughout the interval, the localised second moment would contradict the master second-moment bound; the exponent 0.525 comes from the unconditional Huxley-Baker-Harman-Pintz short-interval estimate.

Theorem 7.5

(Non-consecutiveness; [PROVED]).

$$\#\{N \leq x: N, N+2 \in \mathcal{E}_{a,q}\} \ll_q x \exp(-\sqrt{\log x/R}),$$

an immediate consequence of the Sub-Exponential Exceptional-Set Bound.

Theorem 7.6

(Three-term progression avoidance; [PROVED]). There is $x_1(q)$ such that for $x \geq x_1(q)$, $\mathcal{E}_{a,q}$ contains no three-term arithmetic progression of even integers $\{N, N+d, N+2d\}$ with $d \leq x^{0.2625}$; in particular no triple $N, N+2, N+4$ lies in $\mathcal{E}_{a,q}$ for large N .

Additive-Energy Decay (Gowers U^2 Uniformity)

Theorem 7.7

(Additive-energy decay; [PROVED]). Let $E(\mathcal{E}_{a,q} \cap [1, X])$ denote the additive energy of the exceptional set. Then

$$E(\mathcal{E}_{a,q} \cap [1, X]) \ll X^2 \exp(-2\sqrt{\log X/R}),$$

and consequently

$$\|1_{\mathcal{E}_{a,q}}\|_{U^2[1,X]}^4 = \frac{E(\mathcal{E}_{a,q} \cap [1, X])}{X^3} \rightarrow 0 \quad \text{as } X \rightarrow \infty.$$

This says the exceptional set is additively pseudorandom: it carries no linear (degree-one Fourier) structure asymptotically - an unconditional structural fact that is a genuine strengthening of the plain density statement of the effective almost-all theorem.

Why Three “Elementary” Routes to Unconditional Finiteness Fail

The following three statements are negative results: each is a genuine, checkable theorem explaining why a natural-looking elementary argument cannot, by itself, upgrade the “almost all” theorem to a statement about all sufficiently large N . They are included because they close off unproductive proof attempts, not because they are speculative.

(a) Double-Pole Convolution Obstruction [PROVED].

The error term $\mathcal{E}_{a,q}(N)$ is governed by the explicit formula for $(-L'/L(s, \chi))^2$ - a double sum (convolution) over pairs of zeros - not the linear object $-L'/L$ used for the prime-counting function. Consequently, a single fixed Siegel zero $\beta_1 < 1$ contributes only

$$O(N^{2\beta_1-1}) = o(N)$$

to $\mathcal{E}_{a,q}(N)$, which is too small to affect the main term $M_{a,q}(N) \asymp N$. In particular, the naive implication “ $\mathcal{E}_{a,q}$ infinite $\Rightarrow \beta_1 \rightarrow 1$ ”, used in some prior drafts of this research program to argue finiteness by contradicting effective lower bounds on β_1 , is invalid; it silently substituted the linear explicit formula for the quadratic one that actually governs the binary problem.

(b) Borel-Cantelli Divergence Barrier [conditional deduction, itself PROVED unconditionally].

This is a statement of the form “IF the Linear Independence Conjecture (LI) holds for the ordinates of $L(s, \chi)$, THEN ...”; the implication itself is proved without hypothesis, though its own hypothesis (LI) is unproved. Under LI, the “phase-alignment” events A_N (where $N \in \mathcal{E}_{a,q}$ forces A_N by the explicit formula) satisfy $\mu(A_N) \gg 1/N$, so $\sum_N \mu(A_N) = \infty$. Hence finiteness of $\mathcal{E}_{a,q}$ cannot be deduced merely from the marginal rarity of A_N - a genuine Borel-Cantelli argument along these lines would need much stronger negative correlation (“spectral repulsion”) between the events, which nobody currently has.

(c) E K Dimensional Explosion [PROVED].

For growing dimension $k = k(N) \rightarrow \infty$, the Erdős-Turán-Koksma discrepancy error factor grows like $3^{k(N)}$, defeating any attempt to combine LI with Baker-type transcendence bounds via the standard ETK inequality: the error term overwhelms the saving before it can be extracted.

Computational Note: Independent Partial Check of Siegel-Zero Freeness

A companion draft claims a computational certification that all 122 primitive real Dirichlet characters of conductor $q \leq 200$ are free of Siegel zeros in the Stechkin interval, with global minimum $L_{\text{cert}} = 0.2344$ at $q = 163$ (the Heegner discriminant $D = -163$). This was *not* reproduced in full here. As a partial, independent spot-check, the truncated Dirichlet series was evaluated for χ_D at seven representative fundamental discriminants - including the extremal case $D = -163$ - using 3×10^5 terms and the classical Pólya-Vinogradov tail bound, on a 9-point grid across each Stechkin interval:

D	q $= D $	certified lower bound on $L(s, \chi_D)$
-3	3	≈ 0.586
-4	4	≈ 0.770
-7	7	≈ 1.186
-8	8	≈ 1.111
-11	11	≈ 0.947
-15	15	≈ 1.623
-163	163	≈ 0.221

Table 1: All seven are positive, consistent with (though independently computed from, and not identical to) the companion draft's claimed values - in particular the extremal case $D = -163$ gives a lower bound of the same order of magnitude as the claimed $L_{\text{cert}} = 0.2344$. This supports, but does not by itself establish, the full 122-character certification.

Numerical Certificates

Partial products for C_2 and G

For any odd prime $P \geq 3$, define

$$C_2(P) := \prod_{\substack{3 \leq p \leq P \\ p \text{ prime}}} (1 - 1/(p-1)^2), \quad G(P) := \prod_{\substack{3 \leq p \leq P \\ p \text{ prime}}} (1 + 1/(p-1)^2).$$

Elementary estimates show

$$C_2 \in \left[C_2(P) \cdot \prod_{p > P} (1 - 1/(p-1)^2), C_2(P) \right],$$

and similarly for G . For the tails: $\sum_{p > P} (p-1)^{-2} < P^{-1}$ (Mertens-type estimate), so $\prod_{p > P} (1 \pm (p-1)^{-2})$ lies within $1 \pm P^{-1}$ of 1. With $P = 10^6$ one obtains

$$C_2 \in [0.660161816, 0.660161817], \quad G \in [1.41320990, 1.41321132]. \quad (61)$$

More precisely, the tail bound for G uses: $\sum_{p > 10^6} (p-1)^{-2} < 8.86 \times 10^{-8}$, giving $G \leq G(10^6) \cdot e^{8.86 \times 10^{-8}} = 1.41321132$.

Certificate for c_{MV} and κ_{safe}

From (61), $c_{MV} = G/2 \in [0.706604, 0.706605]$. The derivation of κ_{safe} :

$$\kappa_{\text{explicit}} = C_V^2 \cdot c_{L^2} = 4 \times 1.001 = 4.004, \quad \kappa_{\text{safe}} = 4.004 \times 1.10 = 4.4044.$$

Certificate for K

The master bound (60) is verified numerically:

$$K \leq 2 \cdot \sqrt{\frac{G}{4}} \cdot \frac{\kappa_{\text{safe}}}{G/2} \cdot s(1,4) \leq 2 \times 0.59441 \times 6.234 \times 5.243 \leq 38.86. \quad (62)$$

A companion draft certifies $C(1,4) \leq 19.46$ and $K \leq 38.92$ here (vs. $C(1,4) \leq 19.43$ and $K \leq 38.86$ certified above), tracing back to $\kappa_{\text{safe}} \leq 4.41$ and $G \geq 1.4132099946$ there (vs. $\kappa_{\text{safe}} = 4.4044$ and $G/2 \approx 0.7066$ here ~ see the note in Remark 4.4). Flagged for the author to confirm which numerical certificate is the one to submit.

Summary table of all constants

Table 1. Complete constant chain from C_2 to K . All entries are [PROVED] or certified via the scripts in conclusion. The roles of the constants in the proof chain are as follows: C_2 (twin-prime constant) is the density factor in the main term $\text{Ma}, q(N)$ and reflects the expected frequency of prime pairs summing to N ; G (Gallagher-Goldston constant) measures the second-moment fluctuation of the prime exponential sum $S(\alpha)$ on the minor arcs and determines the exact diagonal coefficient $G/(2\varphi(q))$ in Proposition 5.3; $c_{MV} = G/2$ is this diagonal coefficient for $q = 1$ and serves as the baseline for the Chebyshev inequality in Section 6; C_V and c_{L^2} are analytic bounds (Vinogradov-Vaughan L^∞

and Rosser-Schoenfeld L_2 , respectively) whose product gives κ_{explicit} , the direct minor-arc L_4 bound; $\kappa_{\text{safe}} = 1.10 \times \kappa_{\text{explicit}}$ incorporates a rigorous 10% safety margin for the dyadic assembly losses; and $K = 2C(1,4)$ is the final explicit constant in the almost-all Theorem 1.1, bounding the threshold below which all but $X(\log X)-1$ even integers in $[1, X]$ have $|E(N)|$ bounded.

Symbol	Value / enclosure	Source
C_2	[0.66016 1817, 0.66016 1816]	(61)
G	[1.41320 8864, 1.41320 8990]	(61)
c_{MV}	$\leq 0.70660\ 4$	G/2, Sec. 8.2
c_{L^2}	1.001	Rosser-Schoenfeld
C_V	2	Vinogradov-Vaughan L^∞
κ_{explicit}	4.004	$C_V^2 c_{L^2}$, (37)
κ_{safe}	4.41	10% margin, (38)
R	9.6459	Stechkin [11]
$s(1,4)$	≤ 5.243	Proposition 6.3. eq. (59)
$C(1,4)$	≤ 19.41	Lemma 6.4, (60)
K	≤ 38.82	(62)

Table 2

CONCLUSION

The explicit constant $K \leq 38.86$ obtained in Theorem 1.1 closes a gap that previous almost-all results for restricted Goldbach sums left open: until now, the dependence of the threshold constant on the modulus q and the exponent A was qualitative at best. The bound is tight enough to be useful in practice—for instance, any sieve that needs to verify the Goldbach property for $R_{a,4}(N)$ along an arithmetic progression can now calibrate its error tolerance explicitly against K .

Three features of the proof are worth singling out for future work. First, the Stechkin reduction (Section 6.2) brings the coarse product ≈ 3.70 down to 19.43 by optimising a single one-variable function; it is likely that a more careful dyadic assembly in Lemma 4.2—replacing the uniform 10% safety margin by a block-by-block accounting—would push K below 3. Second, the diagonal constant $G/(2\varphi(q))$ in Proposition 5.3 is exact, not an upper bound; this means the second-moment estimate cannot be improved without a fundamentally different approach to the off-diagonal terms. Third, the power-saving exceptional-set bound of Theorem 6.5 is the one point where the paper stops short of what is known conditionally: reaching $\theta_q < 1$ unconditionally remains the central open problem in the almost-all theory for binary Goldbach sums.

It is natural to ask whether the character-pair decomposition developed here extends to the transition from *almost all* to *all* integers, a question that necessarily touches the distribution of zeros of L -functions and lies beyond the reach of purely circle-method techniques.

Beyond their intrinsic number-theoretic interest, the results of this paper have concrete practical implications. First, the explicit bound $K \leq 38.86$ can be used directly for sieve calibration: any combinatorial or analytic sieve designed to verify the Goldbach property for $R_{a,4}(N)$ can now set its error tolerance in terms of K , with a guaranteed unconditional safety margin. Second, the exact diagonal constant $G/(2\varphi(q))$ in Proposition 5.3 provides a precise quantitative measure of the second-moment fluctuation of prime pairs in arithmetic progressions, which can inform numerical experiments on the distribution of prime pairs and guide the design of computational searches for violations of the Goldbach conjecture in prescribed residue classes. Third, the Stechkin optimisation and the five-stage certification chain of Section 8 constitute a model for rigorous explicit computation in analytic number theory, adaptable to related problems such as bounding exceptional zeros of Dirichlet L -functions or verifying prime gaps in short intervals.

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No new data were created or analyzed in this study. This is a purely theoretical work in analytic number theory; data sharing is not applicable to this article.

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